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STABILITY OF STRUCTURE ELEMENTS SUBJECTED TO STATIONARY LOADS

V. D. Potapov

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The problem of the stability of viscoelastic rods and shells subjected to compressive loads varying randomly with time was examined in [1]. The method of moment functions is used for the solution. The problem mentioned is among the class of stochastically nonlinear problems; hence, the system of equations in the desired moment functions turns out not to be closed [2-4]. Closure of the system of equations is realized by using the hypothesis that the process being studied is quasi-Gaussian, whereupon an approximate solution is obtained. A feature in the construction of such a solution makes estimation of the degree of its error quite problematical in the general case. From this viewpoint, an analysis of the exact solutions of the problems mentioned is of indubitable interest since its illustration can result in a comparison between the outcomes obtained by approximate and exact methods.

This paper is devoted to an examination of the exact method of solving problems on the stability of structure elements subjected to random loads.

We assume that a viscoelastic rod loaded by stationary transverse loads and a compressive force applied to the ends is at rest on a continuous viscoelastic foundation. The equilibrium equation for such a rod in the quasistatic formulation of the problem is

$$w = -(c + \mathbf{K})\,[(1 - \mathbf{\Gamma})EIw^{\mathrm{IV}} + P(w + w_0)'' - q], \tag{1}$$

where

$$\mathbf{\Gamma} f = \int_{t_0}^{t} \Gamma(t-\tau)\, f(\tau)\, d\tau, \quad \mathbf{K}\varphi = \int_{t_0}^{t} K(t-\tau)\,\varphi(\tau)\, d\tau;$$

and w, w_0 are the additional and initial rod deflections. The remaining notation is standard.

The relaxation $\Gamma(t-\tau)$ and creep $K(t-\tau)$ kernels characterize the viscous properties of the material of the rod and of the viscoelastic foundation.

Considering the rod hinge-supported at the ends and assuming

$$w_0(x) = f_0 \sin\frac{k\pi}{l}x,$$
$$w(x, t) = f(t)\sin\frac{k\pi}{l}x, \quad q(x, t) = q^0(t)\sin\frac{k\pi}{l}x,$$

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we obtain from (1)

$$f = -(c + K)\left[(1 - \Gamma) EI \frac{k^4\pi^4}{l^4} f - \frac{k^2\pi^2}{l^2} P(f + f_0) - q^0\right]. \quad (2)$$

Later representing the kernels $\Gamma(t - \tau)$, $K(t - \tau)$ as a linear combination of exponentials, the integral equation (2) can be rewritten as an ordinary differential equation.

Let us write the random functions $P(t)$ and $q^0(t)$ as follows:

$$P(t) = P_0 + P'(t),\ q^0(t) = q_0 + q'(t),$$

where

$$P_0 = \langle P(t)\rangle = \text{const};\ q_0 = \langle q^0(t)\rangle = \text{const}.$$

Considering the stationary processes $P'(t)$, $q'(t)$ as the result of "white" noise passing through linear filters and taking account of the exponential nature of the kernals $K(t - \tau)$, $\Gamma(t - \tau)$, we replace the integral equation (2) by a system of first order ordinary differential equations by expansion of the phase space. The solution of this system of equations is a multidimensional Markov process. The probability density that satisfies the Kolmogorov equation [5] yields the most complete information about this process.

As an illustration permitting the qualitative and quantitative estimation of the behavior of viscoelastic elements compressed by loads that are stationary processes, we consider an elastic rod in a continuous medium, which is a Maxwell body in a rheological sense. Neglecting the elastic strains in the medium as compared with the viscous strains, we write the relationship between the rate of change of the rod deflection and the reaction of the foundation r in the form

$$dw/dt = Ar.$$

We assume that the random component of the compressive load is proportional to the Gaussian "white" noise $\xi(t)$ with the proportionality factor m. Then (2) becomes in differential form (for $t \geqslant t_0 = 0$, $q^0 \equiv 0$)

$$ds/d\tau = -[(1 - \alpha)s - \beta\xi s - f_0]. \quad (3)$$

Here $\tau = \gamma t$ is the new time variable;

$$\gamma = \frac{k^4\pi^4}{l^4} EIA;\quad \beta = \frac{ml^2}{k^2\pi^2 EI};\quad \alpha = \frac{P_0 l^2}{k^2\pi^2 EI};\quad s = f + f_0.$$

We later examine two versions of the problem, $f_0 = 0$ and $f_0 \neq 0$. In the first case we have

$$ds/d\tau = -[(1 - \alpha) - \beta\xi]s. \quad (4)$$

The solution of this homogeneous ordinary differential equation should satisfy the initial condition $s_0 = s(\tau_0)$. The displacement s_0 can be conceived, say, as the cumulative deflection because of the viscous strains of the continuous medium due to loading of the rod by a transverse load prior to the time of application of the longitudinal force (in the absence of a longitudinal force, the deflection s_0 would diminish to zero exponentially with time). If the transverse load is random, the deflection s_0 is a random quantity subject to some distribution law, while if the transverse load is deterministic, then s_0 is a deterministic quantity.

The Fokker–Planck–Kolmogorov equation in the probability density distribution $p(s, \tau, s_0, \tau_0)$ is written as follows:

$$\frac{\partial p}{\partial \tau} = \frac{\partial}{\partial s}[(1 - \alpha) sp] + \frac{1}{2}\frac{\partial^2}{\partial s^2}(\beta^2 s^2 p). \quad (5)$$

The solution of this equation corresponds to the initial condition $p = \delta(s - s_0)$, if s_0 is a deterministic quantity, or $p = p(s_0, \tau_0)$ if s_0 is random, and to the boundary conditions $p \to 0$ as $|s| \to \infty$. Moreover, the function p should satisfy the positivity and normalization conditions.

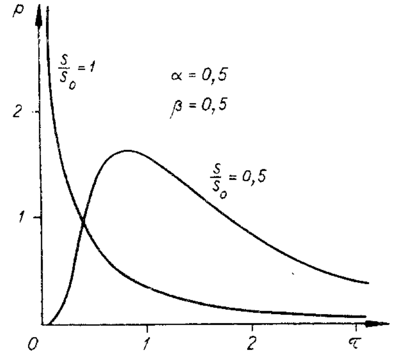

Fig. 1

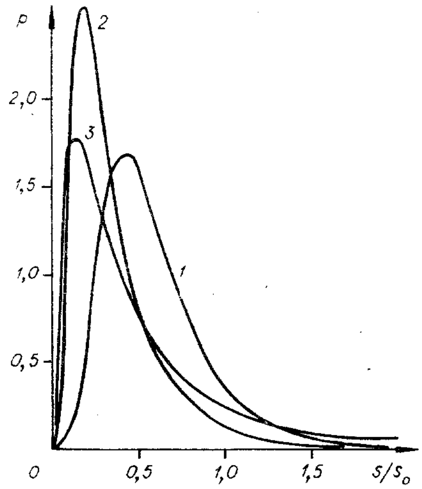

Fig. 2

For instance, if the quantity s_0 is deterministic, then by reducing (5) to a heat conduction equation [6], we obtain (for $\tau_0 = 0$)

$$p(s, \tau, s_0, 0) = \frac{1}{\sqrt{2\pi\beta^2\tau}} e^{(1-\alpha+\beta^2)\tau} \exp\left\{-\frac{[\ln(s/s_0) + (1.5\beta^2 + 1 - \alpha)\tau]^2}{2\beta^2\tau}\right\}.$$

Graphs of the changes in the function p for different values of α, β and τ are represented in Figs. 1 and 2 (the values $\alpha = 0.5$, $\beta = 0.5$, $\tau = 1$; 2) $\alpha = 0.5$, $\beta = 0.5$, $\tau = 2$; 3) $\alpha = 0.5$, $\beta = 1$, $\tau = 1$) correspond to curve 1 in Fig. 2).

By knowing the distribution density $p(s, \tau, s_0)$, the moments of the dimensionless quantities s/s_0 can be determined.

Omitting intermediate computations, we write an expression for the k-th order moments

$$\langle (s/s_0)^k \rangle = \exp\{-[(1-\alpha) - \beta^2(k-1)/2]k\tau\}. \tag{6}$$

The angular brackets here describe the mathematical expectation operation.

The k-th order moment is a damped function of the time if the condition

$$\alpha < 1 - \beta^2(k-1)/2 \tag{7}$$

is satisfied.

For instance, we have for k = 1

$$\alpha < 1. \tag{8}$$

Let us note that for $\beta = 0$ the solution of (4) is asymptotically stable in the Lyapunov sense [7] relative to perturbation of the initial conditions if $\alpha < 1$.

Therefore, the Lyapunov asymptotic stability condition and the damping condition for the mathematical expectation of the rod displacement (or the stability condition in the mathematical expectation of the solution of the initial equation (4) [3, 8]) agree.

For k = 2 we obtain from the relationship (6)

$$\langle (s/s_0)^2 \rangle = \exp[-(2 - 2\alpha - \beta^2)\tau].$$

The solution of (4) is stable in the root-mean-square if

$$\alpha < 1 - \beta^2/2. \tag{9}$$

The stability condition (9) is more rigorous than condition (8). This situation is also confirmed by the general theory of stability of the solutions of the stochastic equations [8]. It follows from the inequality (7) that for a fixed value of the quantities α and β the solution of (4) will be unstable in moments of an order greater than k starting with a certain k.

We later examine the second version of the problem when f_0 is a non-zero deterministic quantity.

Let us write the Fokker—Planck—Kolmogorov equation for this case

$$\frac{\partial p}{\partial \tau} = \frac{\partial}{\partial s}\{[(1-\alpha)s - f_0]p\} + \frac{1}{2}\frac{\partial^2}{\partial s^2}(\beta^2 s^2 p). \quad (10)$$

We limit ourselves to an examination of just the stationary solution by setting $\partial p/\partial \tau \equiv 0$. The general solution of (10) here has the form

$$p\left(\frac{s}{f_0}\right) = C\exp\left(-\frac{2f_0}{\beta^2 s}\right)\left(\frac{s}{f_0}\right)^{-\rho},$$

where $\rho = 2(1 + (1-\alpha)/\beta^2)$.

Let us note that the variable s has the same sign as the initial deflection f_0. The constant C is determined from the normalization condition for the function $p(s/f_0)$. After all the manipulations we arrive at the following expression for the probability density distribution:

$$p\left(\frac{s}{f_0}\right) = \frac{\beta^2}{2\Gamma(\rho-1)}\exp\left(-\frac{2f_0}{\beta^2 s}\right)\left(\frac{2f_0}{\beta^2 s}\right)^{\rho},$$

where $\Gamma(\rho - 1)$ is the Gamma function.

Graphs of the change in the function p(s) as a function of α, β are represented in Fig. 3 (curves 1-4 correspond to the values: 1) $\alpha = 0.5$, $\beta = 0.5$; 2) $\alpha = 0.5$, $\beta = 1$; 3) $\alpha = 1$, $\beta = 1$; 4) $\alpha = 1$, $\beta = 0.5$; and the maximum is reached on curve 4 for $s/f_0 = 4$). It is interesting to note that for the rod loaded by a constant deterministic load for which the parameter α is 1, stationary solutions of the problem turn out to be impossible. Superposition of a random component on the deterministic part of the load results in the appearance of a stationary mode of rod deformation.

A stationary solution of (10) is evidently possible only when

$$\rho - 1 > 0 \quad \text{or} \quad \alpha < 1 + \beta^2/2.$$

The statistical k-th order moments of the ratio s/f_0 are determined by the equalities

$$\left\langle\left(\frac{s}{f_0}\right)^k\right\rangle = \left(\frac{2}{\beta^2}\right)^k \frac{\Gamma(\rho-1-k)}{\Gamma(\rho-1)}.$$

In particular, the mathematical expectation of $\langle(s/f_0)\rangle$ equals

$$\langle(s/f_0)\rangle = 1/(1-\alpha). \quad (11)$$

Let us note that under the action of a deterministic compressive force P_0 the rod displacement s tends to a constant value $f_0/(1-\alpha)$, if $\alpha < 1$ for an unlimited increase in time. Therefore, this condition agrees with the condition of boundedness of the mathematical expectation of the displacement $\langle s \rangle$ in (11) and the stability condition the mathematical expectation (8) of the solution of (4).

Conditions on boundedness of moments of order greater than one, which have the form

$$\rho - 1 - k > 0 \quad \text{or} \quad \alpha < 1 - \beta^2(k-1)/2,$$

evidently also agree with the stability conditions in the moment of the same order for the solution of (4).

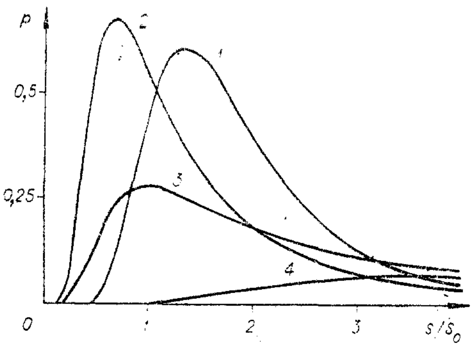

Fig. 3

In conclusion, it should be noted that by using (5) and (10), and for an appropriate selection of the boundary conditions, the problem of determining the probability that the random process s(t) will emerge outside the boundary of a given region can be solved.

The model of an elastic rod located in a continuous viscous medium is a simplified model of a viscoelastic rod whose material possesses limited viscosity. Hence, the results obtained above will apparently agree qualitatively with the analogous results for the mentioned rod compressed by a load, which is a stationary process of the "white" noise type.

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